Lecture 11 (Generating function)

In counting problems, we are often interested in counting the number of objects of 'size n', which we denote by a_n. By varying n, we get different values of a_n. This gives a sequence of real numbers

$$a_0, a_1, a_2, \ldots$$

from which we can define a power sequence

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

The $G(x)$ is called the generating function for the sequence $a_0, a_1, a_2, \ldots$

Consider the problem: We have a die with six faces (numbered 1 to 6) and a die with eight faces (numbered 1 to 8). We roll the dice and we consider the sum of the dice. We want to know the number of ways c_n of getting each number n. We claim that that the generating function $C(x) = \sum_{n \geq 0} c_n x^n$ is given by $C(x) = (x + x^2 + \cdots + x^6) \times (x + x^2 + \cdots + x^8)$. Indeed the first part accounts for the possible outcomes of the first die and the second part accounts for the possible outcome of the second die. For instance getting the sum 5 by getting 2 from the first die and 3 from the second die is accounted by the multiplication of the monomial x^2 from the first parenthesis with monomial from the second parenthesis x^3, etc. Multiplying this out, we get
$C(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 6x^8 + 6x^9 + 5x^{10} + 4x^{11} + 3x^{12} + 2x^{13} + x^{14}$.
In the above problem we see that multiplying generating function is meaningful. Let us now try to generalize the above reasoning. Given two sets, A and B the Cartesian product $A \times B$ is defined as the set of pairs (a, b) with $a \in A$ and $b \in B$. So if A and B are finite the cardinality of these sets are related by $|A \times B| = |A| \times |B|$. We also suppose that the size of a pair (a, b) is the size of a plus the size of b.
For instance, in the example above the class A represents the possible numbers of the first die, so that $A = \{1, 2, 3, 4, 5, 6\}$ and the class B represents the possible number of the second die, so that $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Now $C = A \times B$ represents the possible numbers of the two dice. The size of a number on the first die is just that number, so the generating function for A is $A(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$, while for B is $B(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$. Now the size of a pair of number $(a, b) \in C$ is the sum of the numbers of the two dice. So we want to determine c_n which is the number of pairs (a, b).

1 Techniques for Computation

Let us once again give the definition of a generating function before we proceed.

Definition. Given a sequence $a_0, a_1, a_2, \ldots$, we define the generating function of the sequence $\{a_n\}$ to be the power series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

Let us look at a few examples.

Example 2: Find the generating functions for the following sequences. In each case, try to simplify the answer.

1. $1, 1, 1, 1, 1, 1, 0, 0, 0, 0, \ldots$
2. $1, 1, 1, 1, 1, \ldots$
3. $1, 3, 3, 1, 0, 0, 0, 0, \ldots$
4. ${}^{2005}C_0, {}^{2005}C_1, {}^{2005}C_2, \ldots, {}^{2005}C_{2005}, 0, 0, 0, 0, \ldots$

Solution.
1. The generating function is

$$G(x) = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1 - x^6}{1 - x}$$

2. The generating function is

$$G(x) = 1 + x + x^2 + x^3 + x^4 + \ldots.$$

When $|x| < 1$, $G(x) = \frac{1}{1-x}$.

3. The generating function is

$$G(x) = 1 + 3x + 3x^2 + 1 = (1 + x)^3.$$

4. The generating function is

$$G(x) = {}^{2005}C_0 + {}^{2005}C_1 x + {}^{2005}C_2 x^2 + \ldots + {}^{2005}C_{2005} x^{2005} = (1 + x)^{2005}.$$

Dealing with computations of generating functions, we are particularly interested with two things, namely, whether the generating function can be written in closed form and whether we can find the coefficient of a certain power of x easily. For instance, in the above example, $G(x) = 1 + x + x^2 + x^3 + x^4 + \ldots.$ is not a closed form while $G(x) = \frac{1}{1-x}$ is.

The reason for trying to put a generating function in the closed form is as follows: In the more advanced theory of generating functions, we will find that certain combinations of problems correspond to certain operations (e.g., addition, multiplication, or more complicated operations) on generating functions. If we can find a generating function in closed form, the computations can be greatly simplified and easily carried out.

On the other hand, we are interested in knowing the coefficient of a certain power of x because, as we have remarked at the very beginning, it often refers to the number of objects of size n, which is usually the thing we wish to find in counting problems.

However, if a generating function is given in closed form, ingenious tricks are sometimes required to determine certain coefficients. The following example illustrates some common tricks.

Example 3: For each of the following, find the coefficient of x^{2005} in the $G(X)$.

1. $G(x) = (1-2x)^{5000}$
2. $G(x) = \frac{1}{1+3x}$
3. $G(x) = \frac{1}{(1+5x)^2}$

Solution.
1. By the binomial theorem, we have

$$G(x) = 1 -^{5000} C_1(2x) +^{5000} C_2(2x)^2 - \ldots -^{5000} C_{4999}(2x)^{4999} + (2x)^{5000}.$$

Thus, the coefficient of x^{2005} is $-2^{2005 5000}C_{2005}$.

2. Recalling the formula for the sum to infinity of a geometric series, we have (noting once again that everything is dealt with formally, ignoring questions of convergence)

$$G(x) = \frac{1}{1-(-3x)} = 1 + (-3x) + (-3x)^2 + \ldots$$

Thus, the coefficient of x^{2005} is -3^{2005}.

3. Note that

$$\frac{1}{1+5x} = 1 - 5x + 5^2x^2 - 5^3x^3 + \ldots$$

Hence,

$$G(x) = (1 - 5x + 5^2x^2 - 5^3x^3 + \ldots)(1 - 5x + 5^2x^2 - 5^3x^3 + \ldots).$$

To form an x^{2005} term, we can multiply 1 with $-5^{2005}\,x^{2005}$, $-5x$ with $5^{2004}\,x^{2004}$, 5^2x^2 with $-5^{2003}x^{2003}$ and so on, and finally $-5^{2005}x^{2005}$ with 1. There are altogether 2006 products, each equal to $-5^{2005}x^{2005}$. Thus the coefficient of x^{2005} is -2006×5^{2005}.

The technique used in the above example is rather 'ad-hoc' in nature. It will not work if the power 2 is increased to higher powers. To deal with higher powers, we shall need an extended version of the binomial theorem. For this purpose, we attempt to generalize the binomial theorem for positive integral indices. We begin by extending the usual notion of binomial coefficients to non-integer values.

Definition: For any real number n and positive integer k, we define the extended binomial coefficient ${}^nC_k$ by

$${}^nC_k = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

Also, define ${}^nC_0 = 1$ for any real number n.

Clearly, if n is a positive integer with $n \geq k$, then the above extended binomial coefficient agrees with the usual binomial coefficient. Also, if n and k are positive integers with $n < k$, then we have ${}^nC_0 = 0$. This is natural from a combinatorial point of view: if $n < k$, there is no way to choose k different objects from a collection of n objects. With the notion of the extended binomial coefficients, we can state the extended binomial theorem as follows.

Extended Binomial Theorem: For any real number n, we have

$$(1+x)^n =^n C_0 +^n C_1 x +^n C_2 x^2 + \cdots .$$

Again, if n is a positive integer, we see that the extended binomial theorem agrees with the usual binomial theorem by noting that ${}^nC_k = 0$ when $n < k$.

Since we often have to compute extended binomial coefficients of the form ${}^nC_k$ where n is a negative integer, it is useful to relate the extended binomial coefficients with the usual binomial coefficients. We have the following.

Theorem. For positive integers n and r, we have

$${}^{-n}C_r = (-1)^r \, {}^{n+r-1}C_r.$$

Sometimes, we also need the technique of partial fraction decomposition in the computation, as the following example shows.

Example 4. What is the coefficient of x^{2005} in the generating function $G(x) = \frac{1}{(1-x)^2(1+x)^2}$?

Solution. Let $\frac{1}{(1-x)^2(1+x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2}$.

This gives $A = B = C = D = \frac{1}{4}$.

It follows that

$$G(x) = \frac{1}{4}[(1-x)^{-1} + (1-x)^{-2} + (1-x)^{-1} + (1-x)^{-2}].$$

Thus the coefficient of x^{2005} is $\frac{1}{4}(-^{-1}C_{2005} -^{-2} C_{2005} +^{-1} C_{2005} +^{-2} C_{2005}) = 0$.

Finally, for those who know calculus, the following two examples illustrate some further computation techniques in dealing with generating functions.

Example 5. Find the generating functions of the following sequences in closed form.

1. $1, 2, 3, 4, 5, 6, 7$
2. $0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$

Solution. If $G(x) = a_0 + a_1 x + a_2 x^2 + \cdot$,

Then $G'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$.

The same is true for integration in place of differentiation.

1. Now $G(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{d}{dx}(x + x^2 + x^3 + \cdots) = \frac{d}{dx}(\frac{x}{1-x}) = \frac{1}{(1-x)^2}$.

One can verify the above from binomial expansion also.

2. $G(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \int (1 - x + x^2 - x^3 + \cdots)dx = \int \frac{dx}{1+x} = \ln(1+x) + C$

To find the constant C, we put in $x = 0$ to get $C = G(0) = a_0 = 0$.

Hence the answer is $G(x) = \ln(1+x)$.

Example 5. Find the coefficient of x^{2005} for each of the following generating functions.

1. $G(x) = \ln(1-x)$
2. $G(x) = \sin x$

Solution. 1. We have $G(x) = \int -\frac{1}{1-x}dx = C - x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

where $C = G(0) = 0$. From the above expression, we see that the coefficient of x^{2005} is $-\frac{1}{2005}$.

2. Let $G(x) = a_0 + a_1x + a_2x^2 + \cdots$. Setting $x = 0$, we get $a_0 = 0$.

Now differentiating $G(x)$, we get

$\cos x = a_1 + 2!a_2x + 3!x^2 + \cdots$

Again setting $x = 0$, give $a_2 = 0$. Continuing this process, we see that $-1 = 3!a_3$, $0 = 4!a_4$, $1 = 5!a_5$, $0 = 6!a_6$, $-1 = 6!a_6$, $0 = 8!a_8$, and son on. Since $2005 = 1\,(mod)\,4$, the coefficient of $x^{2005} = \frac{1}{2005!}$.

2 Applications of Generating Functions

We see some elementary applications of generating functions. Consider the following example.

Example 1: Find the coefficient of x^{17} in the expansion of $(1 + x^5 + x^7)^{20}$.

Solution: The only way to form an x^{17} term is to gather two x^5 and one x^7 . Since there are ${}^{20}C_2 = 190$ ways to choose two x^5 from the 20 multiplicands and ${}^{18}C_1 = 18$ ways to choose one x^7 from the remaining 18 multiplicands, the answer is $190 \times 18 = 3420$.

Let us describe the above problem in another way.

Suppose there are 20 bags, each containing a 5 coin and a 7 coin. If we can use at most one coin from each bag, in how many different ways can we pay 17, assuming that all coins are distinguishable (i.e. the 5 coin from the first bag is considered to be different from that in the second bag, and so on)?

It should be quite clear that the answer is again 3420. To pay 17, one must use two 5 coins and one 7 coin. There are ${}^{20}C_2 = 190$ ways to choose two 5 coins from the 20 bags, and 18 ways to choose a 7 coin from the remaining 18 bags.

Using the notations we introduced at the very beginning, we could say that $a_{17} = 3420$.

Some well-known results in combinatorics can be reproduced by means of generating functions, as the following examples show.

Example. There are 50 students in a class. How many ways are there to select 6 students to represent in the International Math Olympiad?

Solution. Each student is either selected or not selected. Hence each student contributes a factor of $1+x$ to the generating function, where the term 1 (i.e. x^0) refers to the case when the student is not selected (i.e. the student occupies 0 place) while the term x (i.e. x^1) refers to the case when the student is selected (i.e. the student occupies 1 place). Since there are 50 students, the generating function is

$$(1+x)^{50}.$$

Since 6 students are to be selected, the answer is the coefficient of x^6 in the $G(x)$, which, according to the binomial theorem, is ${}^{50}C_6$. This, of course, agreed with what we would have obtained without using generating functions.

It is worth noting how the generating function is formed. Basically, it is formed by a sequence of $'+'s$ and $'\times's$, corresponding to a sequence of 'OR's and 'AND's, very much like how counting problems are typically formulated. For each student, he is either selected OR not selected, so each student contributes a factor of $1+x$. Now we need to do the same for the 1st student AND the 2nd student AND the 3rd student AND so on. That's why we multiply 50 copies of $1+x$ together to form the generating function.

Example. There are 30 identical souvenirs, to be distributed among the 50 IMO trainees, and each trainee may get more than one souvenir. How many ways are there to distribute the 30 souvenirs among the 50 trainees?

Solution. Each student may get 0 OR 1 OR 2 OR ... souvenirs, thus contributing a factor of $1+x+x^2+\cdots$. Since there are 50 students, the generating function is

$$G(x)=(1+x+x^2+\cdots)^{50}=(\frac{1}{1-x})^{50}=(1-x)^{-50}$$

As there are 30 souvenirs, the answer is the coefficient of x^{30} in $G(x)$, which, according to the extended binomial theorem, is equal to ${}^{-50}C_{30}={}^{79}C_{30}$.

Note. One may argue that the term $1+x+x^2+\cdots$ should be replaced by $1+x+x^2+\cdots+x^{30}$ in the above generating function because each student may get at most 30 souvenirs. It turns out that this modification will not affect the final outcome, and the details are left as an exercise. In view of this, we will simply use $1+x+x^2+\cdots$ most of the time because it is easier to handle.

Using the same idea employed in the previous examples, we can solve more complicated counting problems using generating functions, as can be seen in the following examples.

Example. How many integer solutions to the equation $a+b+c=6$ satisfying $-1\le a\le 2$ and $1\le b,c\le 4$?

Solution. The generating function is

$$G(x) = (x^{-1} + x^0 + x^1 + x^2)(x^1 + x^2 + x^3 + x^4)^2 = x(1 + x + x^2 + x^3)^3 = x(\frac{1 - x^4}{1 - x})^3$$

$$= x(1 - 3x^4 + 3x^8 - x^{12})(1 - x)^{-3} = (x - 3x^5 + 3x^9 - x^{13})(1 - x)^{-3}.$$

The answer is the coefficient of x^6 in the $G(x)$. To get an x^6 term, we can multiply x with the x^5 term in $(1 - x)^{-3}$, as well as multiply $-3x^5$ with the x term in $(1 - x)^{-3}$. By the extended binomial theorem, the coefficient of x^5 and x in $(1 - x)^{-3}$ are $-^{-3}C_5$ and $-^{-3}C_1$ respectively. Hence the answer is $-^{-3}C_5 - 3(-^{-3}C_1) = 12$.

Example. In a country there are coins of denominations Rs. 2, Rs. 3, Rs. 5, and Rs. 7. How many different ways are there to pay exactly Rs. 10?

Solution. The Rs. 2 coins may contribute a sum of $0, 2, 4, \cdots$, thus leading to the factor $1 + x^2 + x^4 + \cdots$. Using the same idea for the 3, 5 and 7 coins, the generating function is given by

$$G(x) = (1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^7 + x^{14} + \cdots)$$

$$= \frac{1}{1 - x^2} . \frac{1}{1 - x^3} . \frac{1}{1 - x^5} . \frac{1}{1 - x^7}$$

The answer is the coefficient of x^{10} in $G(x)$.

3 Solving Recurrence Relations

Given a recurrence relation, one of the common ways to solve for the general term of a sequence is to use the method of characteristic equations. Here we will see how generating functions may be employed as an alternative way in solving recurrence relations.

For each sequence, we can form a generating function which may be regarded as an 'infinite polynomial'. If a recurrence relation is given, we can possibly reduce this 'infinite polynomial' to a finite one, so that we can get the generating function in closed form. Consequently the coefficient of a general term (i.e. the general term of the sequence) can be found. The idea is more or less the same as that employed in deriving the sum of a geometric series.

Example. Using generating functions, find a_n in terms of n in each of the following cases.

1. $a_0 = 2$ and $a_{n+1} = 3a_n$ for $n \geq 0$
2. $a_0 = 1$, $a_1 = 2$, and $a_{n+2} = 5a_{n+1} - 4a_n$ for $n \geq 0$

Solution. In each case, we let $G(x)$ be the generating function for the given sequence $\{a_n\}$.

1. We have

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots$$

This gives

$$3x \times G(x) = 3a_0x + 3a_1x^2 + 3a_2x^3 + \cdots$$

Then

$$(1-3x)G(x) = a_0 + (a_1 - 3a_0)x + (a_2 - 3a_1)x^2 + \cdots$$

Since $a_0 = 2$ and $a_{n+1} = 3a_n$ for $n \geq 0$, we have $(1-3x)G(x) = 2$, i.e.

$$G(x) = \frac{2}{1-3x} = 2[1 + (3x) + (3x)^2 + (3x)^3 + \cdots].$$

In this way, we see that the coefficient of x^n in $G(x)$ is $a_n = 2.3^n$, so that $a_n = 2.3^n$ for all n, as we would expect since $\{a_n\}$ is in fact a geometric sequence with first term 2 and common ratio 3.

2. Similarly as above, we have

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$-5xG(x) = -5a_0x - 5a_1x^2 - 5a_2x^3 + \cdots$$

$$4x^2G(x) = 4a_0x^2 + 4a_1x^2 + \cdots$$

Adding these three equations and using the initial conditions as well as the given recurrence relation, we get

$$(1 - 5x + 4x^2)G(x) = 1 - 3x.$$

Applying partial fraction decomposition, we have

$$G(x) = \frac{1-3x}{1-5x+4x^2} = \frac{2}{3}\left(\frac{1}{1-x}\right) + \frac{1}{3}\left(\frac{1}{1-4x}\right)$$

$$= \frac{2}{3}[1 + x + x^2 + x^3 + \cdots] + \frac{1}{3}[1 + (4x) + (4x)^2 + (4x)^3 + \cdots].$$

Thus the coefficient of x^n in $G(x)$ is $\frac{2}{3} + \frac{1}{3}.4^n$, so that $a_n = \frac{4^n+2}{3}$.